

THE LOWER CENTRAL SERIES OF A RIGHT-ANGLED ARTIN GROUP

RICHARD D. WADE

ABSTRACT. We give a description of Duchamp and Krob's extension of Magnus' approach to the lower central series of the free group to right-angled Artin groups. We also describe how Lalonde's extension of Lyndon words to the partially-commutative settings gives a simple algorithm to find a basis for consecutive quotients of the lower central series of a RAAG.

1. INTRODUCTION

One can often translate problems concerning Lie groups to the world of Lie algebras. When we linearise a problem our life is much easier: we understand vector spaces and their endomorphisms very well, and we may use our knowledge here to give us information about the underlying Lie group. Given a discrete group G , one may form a Lie \mathbb{Z} -algebra by taking the direct sum $\sum_{i=1}^{\infty} \gamma_i(G)/\gamma_{i+1}(G)$, where $\gamma_i(G)$ is the i th term in the lower central series, and the bracket operation is given by taking commutators in G . In general, this Lie algebra may not give much information about G , but if G is a free group the picture is very nice indeed. The Lie algebra one attains is a *free Lie algebra*, and the structure theory of free Lie algebras not only allows us to obtain information about free groups, but $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ also. This correspondence is well-known, and is covered in detail by Magnus in Chapter 5 of [10]. The aim of this paper is to give a description of the analogous theory for right-angled Artin groups, or RAAGs. These results, as well as many of their proofs, are not new, however we feel that a unified summary of key parts of the papers of Duchamp, Krob, and Lalonde [2, 3, 6, 7, 5] will make a useful reference.

We have attempted to make this work as self contained as possible. In particular, we do not assume any results concerning free Lie algebras, which allows the theory of free Lie algebras and the *partially-commutative free Lie algebras* studied here to be developed in parallel. This comes at the cost of assuming certain facts about the combinatorics of words in RAAGs. We feel that this is a reasonable trade-off.

This algebraic approach to the study of the lower central series of a RAAG has much wider implications than the methods in this paper might suggest. The author used Duchamp and Krob's work in [11] to give strong restrictions on how irreducible higher-rank lattices in semisimple Lie groups can act on right-angled Artin groups, and Linnell, Okun, and Schick used the fact the RAAGs are residually torsion-free nilpotent (also shown by this theory) as part of their proof of the strong Atiyah conjecture for RAAGs [8].

Let Γ be a graph with vertex set V and edge set E . Let ι and τ be the maps that send an edge to its initial and terminal vertices respectively. Throughout this paper we will work with a fixed right-angled Artin group determined by Γ and defined

like so:

$$A_\Gamma = \langle v \in V | [\iota(e), \tau(e)] : e \in E \rangle.$$

We will assume Γ is finite with vertex set v_1, \dots, v_r . The paper is set out as follows: we start with a basic introduction to associative and Lie algebras and recount how, given a *central series* $\mathcal{G} = \{G_i\}_{i=1}^\infty$ of a group, one can build a Lie algebra $L_\mathcal{G}$. This is a generalisation of the construction of the Lie algebra associated to the lower central series mentioned before. It is functorial in the sense that if you have two central filtrations $\mathcal{G} = \{G_i\}$ and $\mathcal{H} = \{H_i\}$ of groups G and H respectively, and $\phi : G \rightarrow H$ is a homomorphism such that $\phi(G_i) \subset H_i$ for all i , then there is an induced algebra homomorphism $L_\mathcal{G} \rightarrow L_\mathcal{H}$.

In Section 3 we build up a host of *partially-commutative* objects associated to a right-angled Artin group. Of central importance is the *free partially-commutative monoid* M , which may be viewed as the monoid of positive elements in A_Γ . We define U to be the free \mathbb{Z} -module on M . U inherits a graded algebra structure, with the grading coming from word length in M . One can extend U to an algebra U^∞ by allowing infinitely many coefficients in a sequence of elements of M to be nonzero. U^∞ behaves very much like an algebra of formal power series. For instance, $1 + \mathbf{v}_i$ is a unit in U^∞ , with inverse

$$(1 + \mathbf{v}_i)^{-1} = 1 - \mathbf{v}_i + \mathbf{v}_i^2 - \mathbf{v}_i^3 + \dots$$

and if we define U^* to be the group of units of U^∞ , the mapping $v_i \mapsto 1 + \mathbf{v}_i$ gives an embedding

$$\mu : A_\Gamma \rightarrow U^*,$$

called the *Magnus map*. We define a sequence of subsets $\mathcal{D} = \{D_i\}_{i=1}^\infty$ of A_Γ by saying that $g \in D_i$ if and only if $\mu(g)$ is of the form:

$$\mu(g) = 1 + \text{elements of } U \text{ of degree } \geq i.$$

Theorem 4.11. *For all k , the set D_k is a subgroup of A_Γ and these subgroups satisfy:*

- (1) \mathcal{D} is a central filtration of A_Γ .
- (2) $D_{k+1} \trianglelefteq D_k$ and D_k/D_{k+1} is a finitely generated free abelian group.
- (3) $\gamma_k(A_\Gamma) \subset D_k$.

As μ is injective $\cap_{k=1}^\infty D_k = \{1\}$, and this fact combined with properties (1) and (2) imply that a right-angled Artin group is residually torsion-free nilpotent. If \mathcal{C} is the central filtration given by the lower central series, then property (3) implies that we have a Lie algebra homomorphism $L_\mathcal{C} \rightarrow L_\mathcal{D}$. We finish our study of the Magnus map by using it to give a new proof of the normal form theorem for words in right-angled Artin groups.

U has an associated Lie algebra $\mathcal{L}(U)$ consisting of the elements of U and bracket operation $[a, b] = ab - ba$. In Section 5, we study the Lie subalgebra L_Γ of $\mathcal{L}(U)$ generated by the set $V = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ by looking at Lalonde's description of the *partially commutative free Lie algebra* determined by the graph Γ [6, 7]. His construction goes as follows. One first defines a subset $LE(M) \subset M$ known as the set of *Lyndon elements* of M . These have a very rigid combinatorial structure. In particular there is a way of assigning a bracketing to each Lyndon element; given a subset $X = \{x_1, \dots, x_r\}$ of a Lie algebra L , this bracketing induces a \mathbb{Z} -module homomorphism $\phi_X : \mathbb{Z}[LE(M)] \rightarrow L$. When $X = V$, the induced map $\phi_V : \mathbb{Z}[LE(M)] \rightarrow L_\Gamma$

is an isomorphism. This gives a basis of L_Γ as a free \mathbb{Z} -module, and allows us to give a universal defining property of L_Γ :

Theorem 5.23. *Let L be a Lie algebra, and suppose that $X = \{x_1, \dots, x_r\}$ is a subset of L that satisfies*

$$[x_i, x_j] = 0 \text{ when } [v_i, v_j] = 1.$$

Then there is a unique algebra homomorphism $\psi_X : L_\Gamma \rightarrow L$ such that

$$\psi_X(\mathbf{v}_i) = x_i \text{ for } 1 \leq i \leq r.$$

We use this in Section 6 to construct a chain of algebra homomorphisms

$$L_\Gamma \rightarrow L_C \rightarrow L_D \rightarrow L_\Gamma$$

and show that the composition of the three maps is the identity on L_Γ . In fact:

Theorem 6.3. *L_Γ , L_C , and L_D are isomorphic as graded Lie algebras. Furthermore, the central filtrations \mathcal{C} and \mathcal{D} are equal, so that $\gamma_k(A_\Gamma) = D_k$ for all $k \geq 1$.*

We are now able to use Lyndon elements and L_Γ to describe the lower central series of A_Γ in more detail. For instance, Proposition 4.11 now implies:

Theorem 6.4. *If $k \in \mathbb{N}$, then $\gamma_k(A_\Gamma)/\gamma_{k+1}(A_\Gamma)$ is free-abelian, and $A_\Gamma/\gamma_k(A_\Gamma)$ is torsion-free nilpotent.*

A wonderful aspect of Magnus' approach to the study of free groups is how nicely the overall structure of his work translates to right-angled Artin groups. An avid reader is encouraged to compare Section 4 of this paper with Section 5.5 of [10]. The statements contained in this paper are adapted to deal with the more general setting of RAAGs, however very little work needs to be done in ensuring the proofs then follow through as well.

The author would like to thank Dawid Kielak for a series of helpful comments.

2. ASSOCIATIVE ALGEBRAS, LIE ALGEBRAS, AND CENTRAL FILTRATIONS

An R -algebra is a (left) R -module A equipped with a bilinear map:

$$\cdot : A \times A \rightarrow A,$$

which we call multiplication. A is *associative* if multiplication is associative, so that for all $a, b, c \in A$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

In this case we may ignore the brackets when multiplying elements, and furthermore we will often write $a \cdot b$ as simply ab . We say that A is Lie algebra if it satisfies the following pair of identities:

$$(1) \quad a \cdot a = 0$$

$$(2) \quad a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b) = 0$$

Equation (2) is called the Jacobi identity. In this case, we usually call multiplication the bracket operation, and use $[a, b]$ rather than $a \cdot b$. Given any associative algebra A we may form a Lie algebra $\mathcal{L}(A)$ consisting of the elements of A with the bracket operation

$$[a, b] = a \cdot b - b \cdot a.$$

Given a subset $S \subset A$ we define $\langle S \rangle$ to be the smallest subalgebra of A containing S . We say that $\langle S \rangle$ is the *subalgebra of A generated by S* , and if $\langle S \rangle = A$ we say that S

generates A . We say that S is a basis of A if A is free as an R -module on the set S . A map $\phi : A \rightarrow B$ is an algebra homomorphism if ϕ respects multiplication and the R -module structure of A and B . If S generates A then ϕ is uniquely determined by where it sends S . If A and B are associative, then a morphism $\phi : A \rightarrow B$ induces a morphism $\mathcal{L}(\phi) : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ (\mathcal{L} is a functor from the category of associative algebras to the category of Lie algebras). If U and V are two subalgebras of A , define the algebra $U.V$ to be the linear span of the set $\{u.v : u \in U, v \in V\}$. We recursively define $U^k = U.U^{k-1}$. We say that an algebra A is *graded* if there exist subspaces A_i of A indexed by \mathbb{N} such that $A = \bigoplus_{i=0}^{\infty} A_i$ and $A_i.A_j \subset A_{i+j}$ for all i, j . If $\bigoplus_{i=0}^{\infty} A_i$ is a grading of an associative algebra A , then $\bigoplus_{i=0}^{\infty} A_i$ is also a grading of the associated Lie algebra $\mathcal{L}(A)$. A homomorphism $\phi : A \rightarrow B$ of algebras is *graded* if A and B are graded and $\phi(A_i) \subset B_i$ for all $i \in \mathbb{N}$.

2.1. Lie algebras from central filtrations. Let G be a group. Let $\mathcal{G} = \{G_k\}_{k \geq 1}$ be a sequence of subgroups of G such that for all k, l :

$$\begin{aligned} \text{(F1)} \quad & G_1 = G, \\ \text{(F2)} \quad & G_{k+1} \leq G_k, \\ \text{(F3)} \quad & [G_k, G_l] \subset G_{k+l}. \end{aligned}$$

We say that \mathcal{G} is a *central filtration*, or a *central series* of G . The above conditions imply that $G_k \leq G$ and $G_{k+1} \leq G_k$ for all k . The main construction in this section is a Lie algebra $L_{\mathcal{G}}$ built out of the consecutive quotients G_k/G_{k+1} of \mathcal{G} .

One example of a central filtration is $\gamma(G) = \{\gamma_k(G)\}_{k \geq 1}$, the lower central series of G . This is defined recursively by $\gamma_1(G) = G$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$. Where it is clear which group we are using, we shall simply write γ_k (or γ) rather than $\gamma_k(G)$ (or $\gamma(G)$).

We shall provide a proof that γ is a central filtration in Proposition 2.3. We first have to take a short detour to look at some commutator identities. We use the convention that for $x, y \in G$ we have $[x, y] = xyx^{-1}y^{-1}$, and for conjugation we write ${}^yx = yxy^{-1}$.

Lemma 2.1. *Let x, y, z be elements of G . Then the following identities hold:*

$$\begin{aligned} (3) \quad & {}^xy = [y, x].x \\ (4) \quad & [xy, z] = {}^x[y, z].[x, z] = [x, [y, z]].[y, z].[x, z], \\ (5) \quad & [x, yz] = [x, y].{}^y[x, z] = [x, y].[y, [x, z]].[x, z], \end{aligned}$$

As well as the Witt–Hall identity:

$$[[x, y], {}^yz].[y, z].{}^zx.[z, x].{}^xy = 1.$$

The reader should be aware that the above equations are different to those that occur in many group theory text books; the commutation and conjugation convention we use is set up for left, rather than right, actions. The Witt–Hall identity implies the following theorem:

Theorem 2.2 (Hall, 1933). *Let X, Y and Z be three normal subgroups of G . Then*

$$[[X, Y], Z] \subset [[Y, Z], X].[Z, X], Y]$$

Proof. If $x \in X, y \in Y$ and $z \in Z$, then as X and Y are normal, we have ${}^zx \in X$ and ${}^xy \in Y$. Therefore the Witt–Hall identity implies that $[[x, y], {}^yz] \in$

$[[Y, Z], X] \cdot [[Z, X], Y]$. As Z is a normal subgroup of G we may replace z with $y^{-1}zy$ to show that $[[x, y], z] \in [[Y, Z], X] \cdot [[Z, X], Y]$, and the result follows. \square

We may now prove the previously promised result:

Proposition 2.3. *The lower central series is a central filtration of G .*

Proof. (F1) holds by definition of the lower central series. The second is standard – a simple induction argument shows that $\gamma_k \leq G$, therefore if $x \in G$ and $y \in \gamma_k$ we have $[x, y] = {}^x y \cdot y^{-1} \in \gamma_k$. As γ_{k+1} is generated by elements of this form, $\gamma_{k+1} \leq \gamma_k$, and (F2) also holds. We are left to show that

$$[\gamma_k, \gamma_l] \subset \gamma_{k+l}$$

for all k, l . We allow l to vary and proceed by induction on k . Note that $[\gamma_1, \gamma_l] = \gamma_{l+1}$ by definition. For the inductive step, suppose that $[\gamma_{k-1}, \gamma_l] = [\gamma_l, \gamma_{k-1}] \subset \gamma_{k+l-1}$ for all l . Then by Theorem 2.2:

$$\begin{aligned} [\gamma_k, \gamma_l] &= [[\gamma_1, \gamma_{k-1}], \gamma_l] \\ &\subset [[\gamma_{k-1}, \gamma_l], \gamma_1] \cdot [[\gamma_l, \gamma_1], \gamma_{k-1}] \\ &\subset [\gamma_{k+l-1}, \gamma_1] \cdot [\gamma_{l+1}, \gamma_{k-1}] \\ &\subset \gamma_{k+l}. \end{aligned}$$

Hence the lower central series satisfies (F1), (F2), and (F3) and is a central filtration of G . \square

The identities (4) and (5) imply that if G has a generating set $\{x_1, \dots, x_r\}$ then any consecutive quotient $\gamma_k(G)/\gamma_{k+1}(G)$ of terms in the lower central series is generated by elements of the form $[x_{i_1}, [x_{i_2}, [\dots [x_{i_{k-1}}, x_{i_k}] \dots]] \cdot \gamma_{k+1}(G)$. In particular:

Proposition 2.4. *If G is generated by $\{x_1, \dots, x_r\}$ then $L_{\gamma(G)}$ is generated by $\{x_1\gamma_1(G), \dots, x_r\gamma_1(G)\}$.*

Now let $\mathcal{G} = \{G_i\}_{i \geq 1}$ be any central filtration of G . Let $L_{\mathcal{G}, i} = G_i/G_{i+1}$. As $[G_i, G_i] \subset G_{2i} \subset G_{i+1}$ each $L_{\mathcal{G}, i}$ is an abelian group, therefore we can form a \mathbb{Z} -module $L_{\mathcal{G}} = \bigoplus_{i=1}^{\infty} L_{\mathcal{G}, i}$. Any element in $L_{\mathcal{G}}$ is of the form $\sum_i x_i G_{i+1}$, where each $x_i \in G_i$ and only finitely many x_i are not equal to the identity. As we are in the abelian setting we will often switch between additive and multiplicative notation. For instance, $nx_i G_{i+1} = x_i^n G_{i+1}$, and $x_i G_{i+1} - y_i G_{i+1} = x_i y_i^{-1} G_{i+1}$ in $L_{\mathcal{G}}$. We write

$$x = y \bmod G_i$$

if $xG_i = yG_i$.

Proposition 2.5. *The bracket operation*

$$\left[\sum_i x_i G_{i+1}, \sum_j y_j G_{j+1} \right] = \sum_{i,j} [x_i, y_j] G_{i+j+1}$$

gives $L_{\mathcal{G}}$ the structure of a graded Lie \mathbb{Z} -algebra.

Proof. We need to show that the bracket operation is well-defined, bilinear, and satisfies the Lie algebra axioms (1) and (2) from Section 2. First note that if $x \in \gamma_i$,

$y \in G_j$, and $z \in G_{i+1}$ then $[x, [z, y]] \in G_{2i+j+1}$ and $[z, y] \in G_{i+j+1}$. Therefore by Equation (4) of Lemma 2.1, we have

$$\begin{aligned} [xz, y] &= [x, [z, y]] \cdot [z, y] \cdot [x, y] \\ &= [x, y] \bmod G_{i+j+1}. \end{aligned}$$

Hence the choice of coset representatives in the left hand side of the product does not affect the bracket operation. Similarly, Equation (5) shows that the choice of coset representatives in the right hand side of the bracket does not affect the bracket operation, so the bracket operation is well-defined. We may also use Equation (4) to show that if $x, y \in G_i$ and $z \in G_j$ then

$$[xy, z] = [x, z] + [y, z] \bmod G_{i+j+1},$$

which gives linearity of the bracket operation in the left hand side, and linearity in the right hand side follows in the same way. For the first of the Lie algebra axioms:

$$\begin{aligned} \left[\sum_i x_i G_{i+1}, \sum_j x_j G_{j+1} \right] &= \sum_{i,j} [x_i, x_j] G_{i+j+1} \\ &= \sum_i [x_i, x_i] G_{2i+1} + \sum_{i \neq j} ([x_i, x_j] + [x_j, x_i]) G_{i+j+1} \\ &= 0 \end{aligned}$$

as $[x_i, x_i] = 1$ and $[x_i, x_j] = [x_j, x_i]^{-1}$ for all i, j . To prove the Jacobi identity, we first note that if $x \in G_i$, $y \in G_j$ and $z \in G_k$ then

$$\begin{aligned} [[x, y], {}^y z] &= [[x, y], [y, z]z] && \text{by (3)} \\ &= [[x, y], [y, z]] \cdot [[y, z], [x, y], z] && \text{by (5)} \\ &= [[x, y], z] \bmod G_{i+j+k+1}, \end{aligned}$$

and similarly we have:

$$\begin{aligned} [[y, z], {}^z x] &= [[y, z], x] \bmod G_{i+j+k+1} \\ [[z, x], {}^x y] &= [[z, x], y] \bmod G_{i+j+k+1}. \end{aligned}$$

The Witt–Hall identity of Lemma 2.1 then implies that

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \bmod G_{i+j+k+1},$$

and the Jacobi identity for a general triple of elements in L_G follows. \square

We finish this Section with a useful observation:

Proposition 2.6. *Let $\mathcal{G} = \{G_i\}$ and $\mathcal{H} = \{H_i\}$ be central filtrations of groups G and H respectively. Let $\phi : G \rightarrow H$ be a homomorphism such that $\phi(G_i) \subset \phi(H_i)$ for all $i \in \mathbb{N}$. Then ϕ induces a graded Lie algebra homomorphism $\Phi : L_{\mathcal{G}} \rightarrow L_{\mathcal{H}}$*

Proof. Define $\Phi(\sum_i x_i G_{i+1}) = \sum_i \phi(x_i) H_{i+1}$. As $\phi(G_i) \subset H_i$, this map is well-defined, and the fact that ϕ is a homomorphism ensures that the map Φ is a graded algebra homomorphism. \square

3. THE CAST

In this section we introduce a host of partially-commutative structures associated with A_{Γ} .

3.1. The monoid M_Γ and algebra U_Γ . Let $W(V)$ be the set of (positive) words in $\{v_1, \dots, v_r\}$. The empty word is denoted by \emptyset or 1. We write $|w|$ to denote the length of a word in $W(V)$. We define $\|w\|$, the *multidegree* of a word $w = v_{p_1}^{e_1} \dots v_{p_k}^{e_k}$ to be the element of \mathbb{N}^r with i th coordinate given by

$$\sum_{p_j=i} e_j.$$

If $w, w' \in W(V)$, we say that $w \leftrightarrow w'$ if there exist $w_1, w_2 \in W(V)$ and $v_i, v_j \in A_\Gamma$ such that $[v_i, v_j] = 1$ and

$$\begin{aligned} w &= w_1 v_i v_j w_2 \\ w' &= w_1 v_j v_i w_2. \end{aligned}$$

We then define an equivalence relation on $W(V)$ by saying that $w \sim w'$ if there exist $w_1, \dots, w_n \in W(V)$ such that

$$w = w_1 \leftrightarrow w_2 \leftrightarrow \dots \leftrightarrow w_n = w'$$

Let $M_\Gamma = W(V)/\sim$. Let \bar{w} be the equivalence class of w under the equivalence relation \sim . If $w_1 \sim w'_1$ and $w_2 \sim w'_2$ then $w_1 w_2 \sim w'_1 w'_2$, therefore multiplication of words in $W(V)$ descends to a multiplication operation on M_Γ , with an identity element given by the equivalence class of the empty word. Similarly, if $w \sim w'$ then $|w| = |w'|$ and $\|w\| = \|w'\|$, so we may define the length and multidegree of an element $m \in M_\Gamma$ to be the respective length and multidegree of a word in $W(V)$ representing m . Length and multidegree are additive with respect to multiplication, so that if $m_1, m_2 \in M_\Gamma$ we have:

$$\begin{aligned} |m_1.m_2| &= |m_1| + |m_2| \\ \|m_1.m_2\| &= \|m_1\| + \|m_2\| \end{aligned}$$

This gives the free \mathbb{Z} -module on M_Γ a graded algebra structure in the following way:

Proposition 3.1. *Let U_Γ be the free \mathbb{Z} -module with a basis given by elements of M_Γ . Let $U_{\Gamma,i}$ be the submodule of U_Γ spanned by the elements of M_Γ of length i . Then $U_\Gamma = \bigoplus_{i=0}^{\infty} U_{\Gamma,i}$, and multiplication in M_Γ gives U_Γ the structure of a graded associative \mathbb{Z} -algebra.*

As we will be keeping the graph Γ fixed throughout, we will refer to M_Γ as M , U_Γ simply as U , and a graded piece as U_i rather than $U_{\Gamma,i}$. We will distinguish elements of U from A_Γ by writing positive words in $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ rather than $\{v_1, \dots, v_r\}$.

3.2. U^∞ , an ideal X , and the group of units U^* . Let U^∞ be the algebra extending U by allowing infinitely many coefficients of a sequence of positive elements to be non-zero. Any element of U^∞ can be written uniquely as a power series $a = \sum_{i=0}^{\infty} a_i$, where a_i is an element of U_i . We say that a_i is the *homogeneous part* of a of degree i , and a_0 is the *constant term* of a . Each a_i is a linear sum of elements of M_i , so is of the form $a_i = \sum_{m \in M_i} \lambda_m m$, where $\lambda_m \in \mathbb{Z}$. If $a = \sum_{i=0}^{\infty} a_i$ and $b = \sum_{i=0}^{\infty} b_i$ then the homogeneous part of $a.b$ of degree i is

$$c_i = \sum_{j=0}^i a_j b_{i-j}.$$

If $a^{(0)}, a^{(1)}, a^{(2)}, \dots$ is a sequence of elements of U^∞ , then the sum $\sum_{n=0}^\infty a^{(n)}$ does not always make sense. However, if the set

$$S_i = \{n : a_i^{(n)} \neq 0\}$$

is finite for all i we define $\sum_{n=0}^\infty a^{(n)}$ to be the element of U^∞ with homogeneous part of degree i equal to

$$\sum_{n \in S_i} a_i^{(n)}.$$

Let X be the ideal of U^∞ generated by $\mathbf{v}_1, \dots, \mathbf{v}_r$. Alternatively X is the set of elements of U^∞ with a trivial constant term. In a similar fashion, X^n is the ideal of U^∞ containing all elements $a \in U^\infty$ such that $a_i = 0$ for all $i < n$.

Let U^* be the group of units of U^∞ . One can show (cf. Proposition 4.2) that $a \in U^*$ if and only if $a = \pm 1 + x$ for some $x \in X$. Note that this is much larger than the group of units of U : there is an embedding of A_Γ into U^* called the *Magnus morphism*, or *Magnus map*.

4. THE MAGNUS MAP

To make U^∞ easier to work with, we'd like to treat it as a (noncommutative) polynomial algebra. Specifically, we'd like to have an idea of 'substitution' of elements of U^∞ 'into other elements of U^∞ '. For instance, given a positive word $w = v_{p_1} \dots v_{p_k}$ in $W(V)$ and Q_1, \dots, Q_r in U^∞ we may define $w(Q_1, \dots, Q_r) = Q_{p_1} \dots Q_{p_k} \in U^\infty$. Suppose that Q_1, \dots, Q_r satisfy

$$(6) \quad Q_i Q_j = Q_j Q_i \text{ for all } i, j \text{ such that } [v_i, v_j] = 1.$$

If w and w' are words such that $w \leftrightarrow w'$ then

$$w(Q_1, \dots, Q_r) = w'(Q_1, \dots, Q_r).$$

It follows that if w and w' represent the same element of M , the above equality also holds. Therefore we may define $m(Q_1, \dots, Q_r) = w(Q_1, \dots, Q_r)$, where w is any word in the equivalence class m . This definition respects multiplication in M , so that for $m_1, m_2 \in M$ we have:

$$(7) \quad m_1(Q_1, \dots, Q_r) m_2(Q_1, \dots, Q_r) = m_1 m_2(Q_1, \dots, Q_r).$$

We can't quite substitute variables in any element of U^∞ with this level of generality; for example it is not possible to set $x = 1$ in

$$1 + x + x^2 + x^3 + \dots.$$

However, as long as Q_1, \dots, Q_r have a trivial constant part (in other words they all lie in the ideal X) this problem does not occur.

Proposition 4.1. *Let Q_1, \dots, Q_r be elements of X which satisfy condition (6). Then the mapping*

$$\mathbf{v}_i \mapsto Q_i$$

may be extended to an algebra morphism $\phi : U^\infty \rightarrow U^\infty$.

Proof. Let $a = \sum_{i=0}^\infty a_i$, with $a_i = \sum_{m \in M_i} \lambda_m m$. We define:

$$\phi(a_i) = \sum_{m \in M_i} \lambda_m m(Q_1, \dots, Q_r).$$

If $|m| = i$ then as $Q_j \in X$ for all j , it follows that $m(Q_1, \dots, Q_r)$ lies in X^i . Therefore the smallest nonzero homogeneous part of $\phi(a_i)$ is of degree at least i . Hence the sum $\phi(a) = \sum_{i=1}^{\infty} \phi(a_i)$ is well defined. It is clear from the definition that ϕ is well-behaved under addition and scalar multiplication. Equation (7) tells us that ϕ also behaves well under multiplication, and is an algebra homomorphism. \square

Such substitutions make our life much easier when computing equation in U^∞ ; this is exemplified by the following three propositions:

Proposition 4.2. *If a is of the form $a = 1 + \sum_{i=1}^{\infty} a_i$, then $a \in U^*$ and*

$$a^{-1} = 1 - (a_1 + a_2 + \dots) + (a_1 + a_2 + \dots)^2 - \dots = 1 + \sum_{i=1}^{\infty} c_i.$$

Here $c_1 = -a_1$ and $c_i = -\sum_{j=0}^{i-1} c_j a_{i-j} = -\sum_{j=1}^i a_j c_{i-j}$ recursively.

Proof. One first checks that if $a = 1 + \mathbf{v}_i$ then the element $a^{-1} = 1 - \mathbf{v}_i + \mathbf{v}_i^2 - \dots$ satisfies $a.a^{-1} = a^{-1}.a = 1$. We then attain the general formula for an element of the form $a = 1 + x$ with $x \in X$ by applying the algebra homomorphism given by Proposition 4.1 under the mapping $\mathbf{v}_i \mapsto x$ for all i . The recursive formula is obtained by equating homogeneous parts in the equation $a^{-1}.a = a.a^{-1} = 1$. \square

Proposition 4.3. *Let $x, y \in X$. Then the following formulas hold:*

$$(8) \quad (1+x)(1+y)(1+x)^{-1} = 1 + y + (xy - yx) \sum_{i=0}^{\infty} (-1)^i x^i$$

$$(9) \quad (1+x)(1+y)(1+x)^{-1}(1+y)^{-1} = 1 + (xy - yx) \sum_{i,j=0}^{\infty} (-1)^{i+j} x^i y^j$$

Proof. As in the proof of Proposition 4.2, we first note that these identities hold for $x = \mathbf{v}_i$ and $y = \mathbf{v}_j$ for any i and j . For the general case, we wish to apply Proposition 4.1. If $xy = yx$ then we may pick any i and j and study the algebra homomorphism induced by the mappings $\mathbf{v}_i \mapsto x$, $\mathbf{v}_j \mapsto y$, and $\mathbf{v}_k \mapsto 0$ when $k \neq i, j$. If $xy \neq yx$ then in particular A_Γ is not abelian: in this case pick vertices v_i and v_j such that $[v_i, v_j] \neq 1$, and use the same map as above. \square

Proposition 4.4. *The mapping $v_i \mapsto 1 + \mathbf{v}_i$ induces a homomorphism $\mu : A_\Gamma \rightarrow U^*$.*

Proof. The mapping $v_i \mapsto 1 + \mathbf{v}_i$ induces a homomorphism $\bar{\mu} : F(V) \rightarrow U^*$. If $[v_i, v_j] = 1$ in A_Γ then $\mathbf{v}_i \mathbf{v}_j - \mathbf{v}_j \mathbf{v}_i = 0$ in U^∞ , therefore by Equation (9), relations in the standard presentation of A_Γ are sent to the identity in U^* , and $\bar{\mu}$ descends to a homomorphism $\mu : A_\Gamma \rightarrow U^*$. \square

μ is the *Magnus map*, and is the central object of study in this section. Our first task is to gain some understanding of the image of a generic element of A_Γ under μ .

Definition 4.5. We say that an element $m \in M$ is *square-free* if there exists no word $w \in W(V)$ representing m and an element $v \in V(\Gamma)$ such that vv occurs as a subword of w .

We will now relate square-free elements of M to reduced words representing elements of A_Γ . (Note that our words representing elements of A_Γ are in $W(V \cup V^{-1})$ rather than just $W(V)$).

Definition 4.6. Let $g \in A_\Gamma$ and suppose that $w = v_{p_1}^{e_1} \cdots v_{p_k}^{e_k}$ is a word representing g . We say that w is *fully reduced* if $e_i \neq 0$ for all i and for all i, j such that $v_{p_i} = v_{p_j}$ there exists $i < l < j$ such that $[v_{p_i}, v_{p_l}] \neq 1$.

We define three moves on the set a words of the form $w = v_{p_1}^{e_1} \cdots v_{p_k}^{e_k}$:

- (M1) Remove $v_{p_i}^{e_i}$ if $e_i = 0$.
- (M2) Replace the subword $v_{p_i}^{e_i} v_{p_{i+1}}^{e_{i+1}}$ with $v_{p_i}^{e_i + e_{i+1}}$ if $p_i = p_{i+1}$.
- (M3) Replace the subword $v_{p_i}^{e_i} v_{p_{i+1}}^{e_{i+1}}$ with $v_{p_{i+1}}^{e_{i+1}} v_{p_i}^{e_i}$ if $[v_{p_i}, v_{p_{i+1}}] = 1$.

Given any word w representing g we may find a fully reduced representative of g by applying a sequence of moves of the form (M1), (M2), and (M3). Moves of type (M3) are called *swaps*. If $w = v_{p_1}^{e_1} v_{p_2}^{e_2} \cdots v_{p_k}^{e_k}$ is fully reduced then $\mathbf{v}_{p_1} \mathbf{v}_{p_2} \cdots \mathbf{v}_{p_k}$ is square-free. The following key lemma shows that we can find this square-free form in the k th homogeneous part of $\mu(g)$. We will use $\mu(g)_i$ to denote the i th homogeneous part of $\mu(g)$.

Lemma 4.7. *Let g be a nontrivial element of A_Γ . There exists $k \in \mathbb{N}$ such that k is the largest integer such that there is a square-free element $m \in M_k$ with nonzero coefficient λ_m in the decomposition of $\mu(g)_k$. This element is unique. Furthermore, if $v_{p_1}^{e_1} v_{p_2}^{e_2} \cdots v_{p_l}^{e_l}$ is a fully reduced representative of g then $l = k$, $\mathbf{v}_{p_1} \cdots \mathbf{v}_{p_l} = m$, and $e_1 \cdots e_l = \lambda_m$.*

Proof. By an induction argument on e_i , we have

$$\mu(v_{p_i}^{e_i}) = 1 + e_i \mathbf{v}_{p_i} + \mathbf{v}_{p_i}^2 u_i$$

for some $u_i \in U^*$. Therefore if $v_{p_1}^{e_1} v_{p_2}^{e_2} \cdots v_{p_k}^{e_k}$ is a fully reduced representative of g , we have:

$$\begin{aligned} \mu(g) &= \mu(v_{p_1}^{e_1}) \mu(v_{p_2}^{e_2}) \cdots \mu(v_{p_k}^{e_k}) \\ &= (1 + e_1 \mathbf{v}_{p_1} + \mathbf{v}_{p_1}^2 u_1) (1 + e_2 \mathbf{v}_{p_2} + \mathbf{v}_{p_2}^2 u_2) \cdots (1 + e_k \mathbf{v}_{p_k} + \mathbf{v}_{p_k}^2 u_k). \end{aligned}$$

In this expansion we see that any positive element occurring with length greater than k must contain $\mathbf{v}_{p_i}^2$ as a subword for some i , and the only element of length k without such a subword is $m = \mathbf{v}_{p_1} \cdots \mathbf{v}_{p_k}$, with coefficient $\lambda_m = e_1 \cdots e_k$. As $\mu(g)$ is independent of the choice of fully reduced representative of g , every fully reduced representative $v_{q_1}^{f_1} \cdots v_{q_l}^{f_l}$ must satisfy $l = k$, with $\mathbf{v}_{q_1} \cdots \mathbf{v}_{q_l} = m$ and $f_1 \cdots f_l = \lambda_m$. \square

We have shown that for every nontrivial $g \in A_\Gamma$ there exists $k > 0$ such that $\mu(g)_k$ is nontrivial.

Corollary 4.8. *The homomorphism $\mu : A_\Gamma \rightarrow U^*$ is injective.*

We may now use μ to study the lower central series of A_Γ .

Definition 4.9. Let $g \in A_\Gamma$. We define the *derivation* $\delta(g)$ of g to be equal to $\mu(g)_k$, where k is the smallest integer ≥ 1 such that $\mu(g)_k \neq 0$. If no such k exists, then $g = 1$ and we define $\delta(g) = 0$.

The derivation satisfies the following properties:

Lemma 4.10. *Let $g, h \in A_\Gamma$ and suppose that $\delta(g) = \mu(g)_k$ and $\delta(h) = \mu(h)_l$.*

- (1) *For all integers n , $\delta(g^n) = n\mu(g)_k$.*
- (2) *If $k < l$ then $\delta(gh) = \delta(hg) = \mu(g)_k$.*
- (3) *If $k = l$ and $\mu(g)_k + \mu(h)_l \neq 0$ then*

$$\delta(gh) = \delta(hg) = \mu(g)_k + \mu(h)_l.$$

- (4) *If $k = l$ and $\mu(g)_k + \mu(h)_l = 0$ then either*

$$gh = 1 \text{ or } \delta(gh) \in X^{k+1}.$$

- (5) *If $\mu(g)_k\mu(h)_l - \mu(h)_l\mu(g)_k \neq 0$ then*

$$\delta([g, h]) = \mu(g)_k\mu(h)_l - \mu(h)_l\mu(g)_k.$$

- (6) *If $\mu(g)_k\mu(h)_l - \mu(h)_l\mu(g)_k = 0$ then either*

$$[g, h] = 0 \text{ or } \delta([g, h]) \in X^{k+l+1}.$$

Proof. Parts (2), (3) and (4) follow from standard properties of multiplication in U^∞ . Part (1) follows from part (3), an induction argument on $n > 0$, and induction on $n < 0$. Parts (5) and (6) follow from Equation (9) in Proposition 4.3. \square

Let $D_k = \{g \in A_\Gamma : \mu(g)_l = 0 \text{ if } 0 < l < k\}$. Alternatively, D_k is the set of elements $g \in A_\Gamma$ such that either $g = 1$ or $\delta(g) \in X^k$.

Proposition 4.11. *For all k , the set D_k is a subgroup of A_Γ and these subgroups satisfy:*

- (1) $\mathcal{D} = \{D_i\}_{i=1}^\infty$ *is a central filtration of A_Γ .*
- (2) D_k/D_{k+1} *is a finitely generated free abelian group.*
- (3) $\gamma_k(A_\Gamma) \subset D_k$.

Proof. Parts (2)–(4) of Lemma 4.10 imply that D_k is a subgroup of A_Γ . By definition, $D_1 = A_\Gamma$ and $D_{k+1} \leq D_k$ for all k . Also, if $g \in D_k$ and $h \in D_l$, then $[g, h] \in D_{k+l}$ by parts (5) and (6) of Lemma 4.10. Therefore $\mathcal{D} = \{D_i\}$ satisfies the requirements (F1), (F2) and (F3) given in Section 2.1 and is a central filtration of A_Γ . For part (2), we define the map $\phi : D_k \rightarrow U_k$ by defining $\phi(g) = \mu(g)_k$. Equivalently:

$$\phi(g) = \begin{cases} \delta(g) & \text{if } \delta(g) = \mu(g)_k \\ 0 & \text{otherwise, when } \delta(g) \in X^{k+1}. \end{cases}$$

Parts (2)–(4) of Lemma 4.10 imply that ϕ is a homomorphism to U_k , with kernel D_{k+1} . Therefore the quotient group D_k/D_{k+1} is isomorphic to a subgroup of U_k . As U_k is a finitely generated free abelian group, so is D_k/D_{k+1} . We prove part (3) by induction on k . First note that $A_\Gamma = D_1 = \gamma_1(A_\Gamma)$. Then $\gamma_k \subset D_k$ implies that $\gamma_{k+1} = [\gamma_1, \gamma_k] \subset [D_1, D_k] \subset D_{k+1}$ by part (1). \square

In particular, Proposition 4.11 tells us that the subgroups D_k form a central filtration of A_Γ . Later we shall show that $D_k = \gamma_k(A_\Gamma)$, however this is much harder work. As $\cap_{i=1}^\infty D_i = \{1\}$, part (3) of Proposition 4.11 implies:

Corollary 4.12. *The intersection $\cap_{i=1}^\infty \gamma_i(A_\Gamma) = \{1\}$.*

We finish this section with a proof of a normal form theorem for elements of A_Γ . This is reasonably well-known; Green's thesis [4], contains a combinatorial proof involving case-by-case analysis. Green's work also extends more generally to graph

products of groups. We give a proof for RAAGs using the Magnus map. The first step is an immediate consequence of Lemma 4.7:

Proposition 4.13. *Let $g \in A_\Gamma$. Let $w = v_{p_1}^{e_1} \cdots v_{p_k}^{e_k}$ and $w' = v_{q_1}^{f_1} \cdots v_{q_l}^{f_l}$ be two fully reduced representatives of g . Then $k = l$.*

In fact, we can prove something much more detailed:

Theorem 4.14. *Let $g \in A_\Gamma$. Let $w = v_{p_1}^{e_1} \cdots v_{p_k}^{e_k}$ and $w' = v_{q_1}^{f_1} \cdots v_{q_k}^{f_k}$ be two fully reduced representatives of g . Then we may obtain w from w' by a sequence of swaps. (Moves of the form $v_{p_i}^{e_i} v_{p_{i+1}}^{e_{i+1}} \mapsto v_{p_{i+1}}^{e_{i+1}} v_{p_i}^{e_i}$ when $[v_{p_i}, v_{p_{i+1}}] = 1$).*

Proof. We proceed by induction on k . We first look at the element $v_{p_1}^{-e_1} g \in A_\Gamma$. Note that $v_{p_2}^{e_2} \cdots v_{p_k}^{e_k}$ and $v_{p_1}^{-e_1} v_{q_1}^{f_1} \cdots v_{q_k}^{f_k}$ are two representatives of $v_{p_1}^{-e_1} g$, and the former representative is fully reduced. Therefore by Proposition 4.13 the latter cannot be fully reduced, so there exists l such that $q_l = p_1$ and $[v_{p_1}, v_{q_i}] = 1$ for $i \leq l$. If $f_l \neq e_1$, then

$$v_{q_1}^{f_1} \cdots v_{q_l}^{f_l - e_1} \cdots v_{q_k}^{f_k}$$

is a fully reduced representative of $v_{p_1}^{-e_1} g$, however this also contradicts Proposition 4.13. Therefore $e_1 = f_l$, and after applying a sequence of swaps to w' we may assume that $v_{p_1} = v_{q_1}$ and $e_1 = f_1$. By induction, $v_{p_2}^{e_2} \cdots v_{p_k}^{e_k}$ may be obtained from $v_{q_2}^{f_2} \cdots v_{q_k}^{f_k}$ by a sequence of swaps, therefore w may be obtained from w' by a sequence of swaps. \square

Given $g \in A_\Gamma$, let $init(g)$ (respectively $term(g)$) be the set of vertices of Γ that can occur as the initial (respectively terminal) letter of a fully reduced word representing g . We say that g is *positive* if $g = 1$ or g can be written as a product $v_1^{e_1} \cdots v_k^{e_k}$ with $e_i > 0$ for all i . As any two fully reduced representatives may be obtained from each other by a sequence of swaps, we have the following immediate corollaries:

Corollary 4.15. *For any $g \in A_\Gamma \setminus \{1\}$, the sets $init(g)$ and $term(g)$ form cliques in Γ : any pair of vertices in $init(g)$ or $term(g)$ commute.*

Corollary 4.16. *The monoid M is isomorphic to the set of positive elements of A_Γ under multiplication.*

5. LYNDON ELEMENTS OF M

In this section we will study the Lie subalgebra of $\mathcal{L}(U)$ generated by the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, which we call L_Γ . The approach is as follows: we first introduce a subset of M called the set of *Lyndon elements*, $LE(M)$. We describe a method for supplying each Lyndon element with a bracketing. If L is a Lie algebra and $X = \{x_1, \dots, x_r\} \subset L$ then this bracketing induces a homomorphism (as \mathbb{Z} -modules) $\phi_X : \mathbb{Z}[LE(M)] \rightarrow L$. In the case that $X = \{\mathbf{v}_1, \dots, \mathbf{v}_r\} \subset L_\Gamma$ we call this induced homomorphism ℓ , and show that ℓ is bijective. Thus we obtain a basis of L_Γ in terms of bracketed Lyndon elements. If $X \subset L$ satisfies

$$[x_i, x_j] = 0 \text{ if } [v_i, v_j]$$

then we show that $\phi_X \ell^{-1} : L_\Gamma \rightarrow L$ is an algebra homomorphism taking \mathbf{v}_i to x_i . This property will then be used in the next section to show that L_Γ and the lower central series algebra of A_Γ are isomorphic.

We deviate here from the approach in Magnus, and instead follow the paper of Lalonde [6]. The analogous free group version is contained in Chapter 5 of [9], and we must start in this world. We first define a lexicographic order on $W(V)$:

Definition 5.1. The *lexicographic ordering* on $W(V)$ is the unique total order $<$ on $W(V)$ that satisfies the following:

- (1) For any nonempty word w , we have $\emptyset < w$.
- (2) If w_1 and w_2 are distinct nonempty words and $x, y \in W(V)$ such that $w_1 = v_i x$ and $w_2 = v_j y$, then $w_1 < w_2$ if either
 - (a) $i < j$ or:
 - (b) $i = j$ and $x < y$.

In particular, $\emptyset < v_1 < v_2 < \dots < v_r$. We state two basic properties of this order:

Lemma 5.2. Let $x, y, z \in W(V)$.

- if $y < z$ then $xy < xz$
- if $|x| \geq |y|$ and $x < y$ then $xz < yz$

The above lemma remains valid if we replace all occurrences of strong inequalities with weak inequalities. The natural projection $\pi : W(V) \rightarrow M$, when coupled with the ordering of $W(V)$, gives us a way of choosing a representative in $W(V)$ for each $m \in M$:

Definition 5.3. Let $m \in M$. Then we define $std(m) \in W(V)$, the *standard representative* of m to be the largest element of $\pi^{-1}\{m\}$ with respect to the lexicographic order.

We then define a total order on M as follows: if $a, b \in M$ we say

$$a < b \text{ if and only if } std(a) < std(b).$$

In view of Lemma 5.2, the following is true:

Lemma 5.4. Let $a, b, c \in M$

- $std(ab) \geq std(a)std(b) \geq std(a)$
- If $b < c$ then $std(a)std(b) < std(a)std(c)$
- If $|a| \geq |b|$ and $a < b$, then $std(a)std(c) < std(b)std(c)$

5.1. Lyndon words. We now describe the notion of *Lyndon words*. These were first introduced by Chen, Fox, and Lyndon in [1]. In this paper, the authors show that in the free group case, the groups D_n introduced in the last section are equal to the lower central series of F_n , and they give an algorithm to determine a presentation of a consecutive quotient γ_n/γ_{n+1} of the lower central series for any finitely presented group. This algorithm is quite complicated, however we shall use the notion of *Lyndon elements* in M , introduced by Lalonde in [6], to give a simple algorithm to describe γ_n/γ_{n+1} in a given right-angled Artin group. Chen, Fox, and Lyndon also relate coefficients of elements in $\mu(g)$ to *Fox derivatives*. Unfortunately these have no natural analog in the partially commutative setting.

We say that w_1 and w_2 are *conjugate* in $W(V)$ if there exist $x, y \in W(V)$ such that $w_1 = xy$ and $w_2 = yx$. Alternatively, w_1 and w_2 are conjugate if they are conjugates in F_n in the usual sense, where $W(V)$ is viewed as a subset of F_n . The *conjugacy class* of w in $W(V)$ is the set of all elements conjugate to w in $W(V)$. A word w is *primitive* if there does not exist $x, y \in W(V) \setminus \{\emptyset\}$ such that $w = xy = yx$.

Definition 5.5. $w \in W(V)$ is a *Lyndon word* if it is nontrivial, primitive and minimal with respect to the lexicographic ordering in its conjugacy class.

Example 5.6. If $V = \{v_1, v_2, v_3, v_4\}$ then v_i is a Lyndon word for all i , and $v_1v_2v_1v_3$ and $v_1v_1v_2$ are Lyndon words. v_1v_1 is not a Lyndon word as it is not primitive, and $v_1v_3v_1v_2$ is not a Lyndon word as it is not minimal in its conjugacy class ($v_1v_2v_1v_3$ is).

There is an assortment of equivalent definitions of Lyndon words.

Theorem 5.7 ([1], Theorem 1.4). *Let $w \in W(V)$. The following are equivalent:*

- (1) w is a Lyndon word.
- (2) For all $x, y \in W(V) \setminus \{\emptyset\}$ such that $w = xy$, $w < y$.
- (3) Either $w = v_i$ for some i or there exist Lyndon words x and y with $x < y$ such that $w = xy$.

The third of these characterisations is particularly appealing, as it allows one to build up a list of Lyndon words recursively.

Example 5.8. If $V = \{v_1, v_2, v_3\}$, then the Lyndon words of length less than or equal to 3 are:

$v_1, v_2, v_3,$
 $v_1v_2, v_1v_3, v_2v_3,$
 $v_1v_1v_2, v_1v_1v_3, v_1v_2v_3, v_2v_2v_3, v_1v_2v_2, v_1v_3v_3, v_2v_3v_3.$

Note that the decomposition of a Lyndon word of length > 1 as a product of two smaller Lyndon words assured to us by part (3) of Theorem 5.7 is not always unique. In this example $v_1v_2v_3$ may be decomposed as $v_1.v_2v_3$ and $v_1v_2.v_3$.

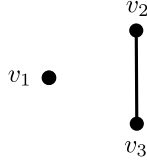


FIGURE 1. A small example graph Γ .

5.2. Lyndon elements. Lyndon elements are the natural generalisations of Lyndon words to the partially commutative setting. Defining conjugation here is more tricky. We first say that two elements m_1, m_2 of M are *transposed* if there exist $x, y \in M$ such that $m_1 = xy$ and $m_2 = yx$. Unfortunately transposition is not an equivalence relation; if Γ is the graph shown in Figure 1, then

$$v_2v_1v_3 \leftrightarrow_{trans.} v_1v_3v_2 = v_1v_2v_3 \leftrightarrow_{trans.} v_3v_1v_2,$$

however $v_3v_1v_2$ cannot be obtained from $v_2v_1v_3$ by a single transposition. We therefore say two elements of M are conjugate if one can be obtained from the other by a sequence of transpositions. Equivalently, two elements are conjugate in M if and only if they are conjugate in A_Γ in the group theoretic sense (when M is viewed as a subset of A_Γ). The set of all elements in M conjugate to m is its *conjugacy class*. We say that m is primitive if there do not exist nontrivial x and y in M such that $m = xy = yx$.

Definition 5.9. $m \in M$ is a *Lyndon element* if it is nontrivial, primitive, and minimal with respect to the ordering of M in its conjugacy class.

Given $g \in A_\Gamma$, we remind the reader that $\text{init}(g)$ is the set of vertices that can appear as the initial letter in reduced words representing g .

Proposition 5.10 ([5], Corollary 3.2). *If m is a Lyndon element, then $\text{init}(m)$ is a single vertex.*

Given $m \in M$, we say that $v_i \in \zeta(m)$ if either $v_i \in \text{supp}(m)$ or there exists $v_j \in \text{supp}(m)$ such that $[v_i, v_j] \neq 1$. Equivalently $v_i \in \zeta(m)$ if and only if either $v_i \in \text{supp}(m)$ or $v_i m \neq m v_i$. In a similar fashion to Lyndon words, there is a selection of equivalent definitions of Lyndon elements.

Theorem 5.11 ([5], Propositions 3.5, 3.6, and 3.7). *Let $m \in M$. The following are equivalent.*

- (1) m is a Lyndon element.
- (2) For all $x, y \in M \setminus \{1\}$ such that $m = xy$, $m < y$.
- (3) Either $|m| = 1$ or there exist Lyndon elements x, y such that $x < y$, $\text{init}(y) \in \zeta(x)$ and $m = xy$.
- (4) $\text{std}(m)$ is a Lyndon word.

Once again, the third part of the classification gives a simple recursive process for writing down Lyndon elements.

Example 5.12. If Γ is the small example graph of Figure 1, then the Lyndon elements of length ≤ 3 are:

$$\begin{aligned} &v_1, v_2, v_3 \\ &v_1 v_2, v_1 v_3 \\ &v_1 v_1 v_2, v_1 v_1 v_3, v_1 v_2 v_2, v_1 v_2 v_3, v_1 v_3 v_3 \end{aligned}$$

The words given here are a subset of the set of Lyndon words on $\{v_1, v_2, v_3\}$. So for example, by condition (3) of Theorem 5.11, the element $v_2 v_3$ does not appear in this list as $v_3 \notin \zeta(v_2) = \{v_1, v_2\}$. As with Lyndon words, the decomposition of a Lyndon element of length > 1 as a product of Lyndon elements is not necessarily unique. In this example $v_1 v_2 v_3$ has two possible decompositions as $v_1 v_2 \cdot v_3$ and $v_1 v_3 \cdot v_2$.

5.3. Standard factorisation. We now give each Lyndon element a unique ‘bracketing’. If m is a Lyndon element of length greater than 1, there may exist many pairs of Lyndon elements x and y such that $m = xy$. If y is minimal in the lexicographic ordering out of all such pairs, we say that $S(m) = (x, y)$ is the *standard factorisation* of m . The standard factorisation behaves well with respect to standard decompositions:

Theorem 5.13 ([7], Proposition 2.1.10). *If $S(a) = (x, y)$ is the standard factorisation of a , then $\text{std}(a) = \text{std}(x)\text{std}(y)$.*

By part 4 of Theorem 5.11, if $S(a) = (x, y)$ then $\text{std}(a) = \text{std}(x)\text{std}(y)$ is a Lyndon word, and is strictly less than its nontrivial conjugates, hence

$$\text{std}(x)\text{std}(y) < \text{std}(y)\text{std}(x).$$

We shall use this trick repeatedly in the work that follows. There is one final combinatorial fact we need before we can move on:

Theorem 5.14 ([7], Proposition 2.3.9). *If a and b are Lyndon elements with $a < b$ and $\text{init}(b) \in \zeta(a)$, then $S(ab) = (a, b)$ if and only if $|a| = 1$ or $S(a) = (x, y)$ and $y \geq b$.*

Example 5.15. We now have a recursive way of giving a bracketing to any Lyndon element. Given $m \in M$, take its standard factorisation $S(m) = (a, b)$, and define the bracketing on m to be equal to $[[a], [b]]$, where $[\cdot]$ denotes the bracketing on a and b respectively. In our small example graph, the only interesting case is $\text{std}(v_1 v_2 v_3) = v_1 v_3 v_2 = \text{std}(v_1 v_3) \text{std}(v_2)$. We then obtain the following bracketing on Lyndon elements of length 3:

$$[v_1, [v_1, v_2]], [v_1, [v_1, v_3]], [[v_1, v_2], v_2], [[v_1, v_3], v_2], [[v_1, v_3], v_3]$$

5.4. Basis Theorem. Let $LE(M)$ be the set of Lyndon elements of M . Let $\mathbb{Z}[LE(M)]$ be the free \mathbb{Z} -module with basis $LE(M)$.

Definition 5.16. Let L be a Lie algebra, and let $X = \{x_1, x_2, \dots, x_r\}$ be elements of L . Let $\phi_X : \mathbb{Z}[LE(M)] \rightarrow L$ be the \mathbb{Z} -module homomorphism defined recursively as follows:

$$\begin{aligned} \phi_X(\mathbf{v}_i) &= x_i & \text{for all } i \\ \phi_X(a) &= [\phi_X(x), \phi_X(y)] & \text{if } |a| > 1 \text{ and } S(a) = (x, y). \end{aligned}$$

Example 5.17. Let L_Γ be the Lie subalgebra of $\mathcal{L}(U)$ generated by the set $V = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. We attain a \mathbb{Z} -module homomorphism $\phi_V : \mathbb{Z}[LE(M)] \rightarrow L_\Gamma$. We write $\phi_V = \ell$. The map ℓ can be thought of as the bracketing procedure for Lyndon elements described above.

Proposition 5.18. *For each $a \in LE(M)$, there exists a subset $I \subset M$ and a set of nonzero integers $\{\alpha_b\}_{b \in I}$ indexed by I such that*

$$\ell(a) = \sum_{b \in I} \alpha_b b.$$

Furthermore, $a \in I$ with $\alpha_a = 1$, and for all $b \in I$ we have $\|b\| = \|a\|$ and $b \geq a$.

Proof. We proceed by induction on $|a|$. If $|a| = 1$ then $\ell(a) = a$ and we are done. Suppose that $|a| \geq 1$. Let $S(a) = (x, y)$ be the standard decomposition of a . By our inductive hypothesis we may write

$$\ell(x) = \sum_{b \in I} \alpha_b b \text{ and } \ell(y) = \sum_{c \in J} \beta_c c$$

with $b \geq x$, $c \geq y$ and $\alpha_x = \beta_y = 1$. Then

$$\begin{aligned} \ell(a) &= [\ell(x), \ell(y)] \\ &= \sum_{b \in I, c \in J} \alpha_b \beta_c (bc - cb) \end{aligned}$$

Then $\|bc\| = \|cb\| = \|xy\| = \|a\|$. Also,

$$\begin{aligned} \text{std}(cb) &\geq \text{std}(c) \text{std}(b) \\ &\geq \text{std}(y) \text{std}(x) \\ &> \text{std}(x) \text{std}(y) \\ &= \text{std}(a). \end{aligned}$$

Hence $cb > a$ for all $b \in I, c \in J$. If either $b > x$ or $c > y$ then

$$\begin{aligned} std(bc) &\geq std(b)std(c) \\ &> std(x)std(y) \\ &= std(a), \end{aligned}$$

Therefore $bc \geq a$ for all b, c . Also, $cb > a$ and $bc = a$ if and only if $b = x$ and $c = y$, so the coefficient of a in the above sum is 1. \square

A consequence of the above theorem is that the image of $LE(M)$ under ℓ forms a linearly independent set.

Corollary 5.19. *The map $\ell : \mathbb{Z}[LE(M)] \rightarrow L_\Gamma$ is injective.*

We now go back to the more general situation.

Lemma 5.20. *Let L be a Lie algebra, and suppose that $X = \{x_1, \dots, x_r\}$ is a subset of L that satisfies*

$$[x_i, x_j] = 0 \text{ when } [v_i, v_j] = 1.$$

Suppose that a is a Lyndon element of M , and $v_i \in V$ such that $[\mathbf{v}_i, a] = 0$ in U . If ϕ_X is defined as in Definition 5.16, then

$$[\phi_X(a), \phi_X(\mathbf{v}_i)] = 0.$$

Proof. We induct on the length of a . If $a = \mathbf{v}_j$ for some j then $[v_i, v_j] = 1$. Therefore $[\phi_X(a), \phi_X(v_i)] = [x_j, x_i] = 0$. If $|a| > 1$ then $S(a) = (x, y)$ for some $x, y \in LE(M)$ such that $[x, \mathbf{v}_j] = [y, \mathbf{v}_j] = 0$. Therefore by induction $[\phi_X(\mathbf{v}_i), \phi_X(x)] = [\phi_X(y), \phi_X(\mathbf{v}_i)] = 0$, and by the Jacobi identity in L :

$$\begin{aligned} [\phi_X(a), \phi_X(\mathbf{v}_i)] &= [[\phi_X(x), \phi_X(y)], \phi_X(\mathbf{v}_i)] \\ &= -[[\phi_X(\mathbf{v}_i), \phi_X(x)], \phi_X(y)] - [[\phi_X(y), \phi_X(\mathbf{v}_i)], \phi_X(x)] \\ &= -[0, \phi_X(y)] - [0, \phi_X(x)] \\ &= 0 \end{aligned} \quad \square$$

What follows is the main technical theorem of this section, which will allow us to extend the \mathbb{Z} -module homomorphism ϕ_X to something that behaves well with respect to brackets also.

Proposition 5.21. *Let L be a Lie algebra, and suppose that $X = \{x_1, \dots, x_r\}$ is a subset of L that satisfies*

$$[x_i, x_j] = 0 \text{ if } [v_i, v_j] = 1.$$

Let ϕ_X be the homomorphism defined in Definition 5.16. Let $a, b \in LE(M)$ be such that $a < b$. Then there exists a subset $I_{a,b} \subset LE(M)$ and a set of integers $\{\alpha_c\}_{c \in I_{a,b}}$ indexed by $I_{a,b}$ such that

$$[\phi_X(a), \phi_X(b)] = \sum_{c \in I_{a,b}} \alpha_c \phi_X(c).$$

Furthermore, each $c \in I_{a,b}$ satisfies the following:

- $c < b$,
- $std(c) \geq std(a)std(b)$,
- $\|c\| = \|ab\|$,

and the sets $I_{a,b}$ and $\{\alpha_c\}_{c \in I_{a,b}}$ are independent of L and X .

Proof. The first step is to define an order \prec on the set of pairs $(a, b) \in LE(M) \times LE(M)$ satisfying $a < b$. We say $(a, b) \prec (a', b')$ if

- $|ab| < |a'b'|$, or
- $|ab| = |a'b'|$ and $std(a)std(b) > std(a')std(b')$, or
- $std(a)std(b) = std(a')std(b')$ and $b < b'$.

Note that the second criterion is possibly the reverse of what one might expect. We shall prove Proposition 5.21 by using induction on the order given by \prec . We drop the subscript of ϕ_X for the remainder of this proof. The base case is when $(a, b) = (v_{r-1}, v_r)$ and is trivial. The inductive step splits into two cases.

Case 1. $init(b) \in \zeta(a)$.

If $|a| = 1$, then Theorem 5.14 tells us $S(ab) = (a, b)$, and $[\phi(a), \phi(b)] = \phi(ab)$ by definition. Also, $ab < b$ by part 2 of Theorem 5.11, and $std(ab) \geq std(a)std(b)$.

If $|a| > 1$, let $S(a) = (x, y)$. This now splits into two subcases.

Subcase 1. $y \geq b$. By Theorem 5.14, we have $S(ab) = (a, b)$, and we are in exactly the same situation as case 1.

Subcase 2. $y < b$. We use the Jacobi identity in L:

$$\begin{aligned} [\phi(a), \phi(b)] &= [[\phi(x), \phi(y)], \phi(b)] \\ &= -[[\phi(b), \phi(x)], \phi(y)] - [[\phi(y), \phi(b)], \phi(x)] \\ &= [[\phi(x), \phi(b)], \phi(y)] + [\phi(x), [\phi(y), \phi(b)]] \end{aligned}$$

We look at the two parts of this sum separately. For $[[\phi(x), \phi(b)], \phi(y)]$, note that $x < a < b$, and $|xb| < |ab|$, so we have $(x, b) \prec (a, b)$. Therefore we may write

$$[\phi(x), \phi(b)] = \sum_{c \in I_{x,b}} \alpha_c \phi(c)$$

with each c satisfying the required criteria. Then for each c , if $y < c$ then

$$\begin{aligned} std(y)std(c) &\geq std(y)std(x)std(b) \\ &> std(x)std(y)std(b) \\ &= std(a)std(b), \end{aligned}$$

so that $(y, c) \prec (a, b)$. If $y = c$ then $[\phi(y), \phi(c)] = 0$. If $c < y$ then as $std(c) = std(x)std(b)$ and $std(y) < std(b)$ we have:

$$\begin{aligned} std(c)std(y) &\geq std(x)std(b)std(y) \\ &> std(x)std(y)std(b) \\ &= std(a)std(b) \end{aligned}$$

so that $(c, y) \prec (a, b)$. In any case, we obtain a decomposition

$$[\phi(c), \phi(y)] = \sum_{d \in I_{c,y}} \beta_d \phi(d)$$

with each d satisfying the required criteria. Furthermore,

$$[[\phi(x), \phi(b)], \phi(y)] = \sum_{c \in I_{x,b}} \sum_{d \in I_{c,y}} \alpha_c \beta_d \phi(d).$$

For the second term in the above sum, $[\phi(x), [\phi(y), \phi(b)]]$, since $y < b$ and $|yb| < |ab|$ we have $[\phi(y), \phi(b)] = \sum_{c \in I_{y,b}} \alpha_c c$ with $c < b$, and

$$\begin{aligned} std(c) &\geq std(y)std(b) \\ &\geq std(y) \\ &> std(x), \end{aligned}$$

so that $x < c$. Also

$$\begin{aligned} std(x)std(c) &\geq std(x)std(y)std(b) \\ &= std(a)std(b). \end{aligned}$$

Hence $(x, c) \prec (a, b)$, and by induction we have

$$[\phi(x), \phi(c)] = \sum_{d \in I_{x,c}} \beta_d \phi(d)$$

with $d < c < b$ and each d satisfies the required criteria. Furthermore

$$[\phi(x), [\phi(y), \phi(b)]] = \sum_{c \in I_{y,b}} \sum_{d \in I_{x,c}} \alpha_c \beta_d \phi(d)$$

The two parts in the equation give us by the Jacobi identity therefore give the required decomposition of $[\phi(a), \phi(b)]$.

Case 2. $init(b) \notin \zeta(a)$.

If $|b| = 1$ then $[\phi(a), \phi(b)] = 0$ by Lemma 5.20, and we are done. If $|b| > 1$, then we write $S(b) = (x, y)$. By the Jacobi identity in L :

$$\begin{aligned} [\phi(a), \phi(b)] &= [\phi(a), [\phi(x), \phi(y)]] \\ &= -[\phi(x), [\phi(a), \phi(y)]] - [\phi(y), [\phi(x), \phi(a)]] \\ &= [[\phi(a), \phi(y)], \phi(x)] - [[\phi(a), \phi(x)], \phi(y)]. \end{aligned}$$

Again we look at the two separate parts in this sum. For $[[\phi(a), \phi(y)], \phi(x)]$, as $(a, y) \prec (a, b)$ we have

$$[\phi(a), \phi(y)] = \sum_{c \in I_{a,y}} \alpha_c \phi(c),$$

with $\|c\| = \|ay\|$ and $std(c) \geq std(x)std(y)$. Note that the smallest letter (with respect to the ordering $v_1 < v_2 < \dots < v_r$) of any Lyndon word must be its initial letter, otherwise there would be a conjugate of that word that is smaller with respect to the ordering of M . Let $inf(g)$ denote the smallest letter in $supp(g)$ for any $g \in M$. As $\|c\| = \|ay\|$, we have:

$$init(c) = inf(c) = inf(ay) \leq inf(a) = init(a) < init(b) = init(x).$$

The strict inequality holds in the above as $a < b$ and $init(a) \neq init(b)$ because $init(b) \notin \zeta(a)$. Hence $c < x$, and

$$\begin{aligned} std(c)std(x) &\geq std(a)std(y)std(x) \\ &> std(a)std(x)std(y) \\ &= std(a)std(b) \end{aligned}$$

Therefore $(c, x) \prec (a, b)$, and there is a decomposition

$$[\phi(c), \phi(x)] = \sum_{d \in I_{c,x}} \beta_d \phi(d),$$

with each d satisfying the required criteria; the only bit being tricky is the fact that $d < b$. For this we use the same method as above – $\text{init}(d) \leq \text{init}(a) < \text{init}(b)$ and therefore $d < b$. For $[[\phi(a), \phi(x)], \phi(y)]$ the same method applies and we will spare the reader any further details.

This completes the induction proof. The only part we have not covered is the fact that the sets $I_{a,b}$ and $\{\alpha_c\}_{c \in I_{a,b}}$ are independent of X and L , however this is clear as we did not need use our choice of L or X at any point in the proof. \square

Proposition 5.21 implies that the image of ℓ in L_Γ is closed under the bracket operation, so is a subalgebra of L_Γ . As L_Γ is the smallest subalgebra of $\mathcal{L}(U)$ containing $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, and this set is in the image of ℓ , this means that ℓ is surjective. We have shown in Corollary 5.19 that ℓ is also injective.

Corollary 5.22. *The map $\ell : \mathbb{Z}[LE(M)] \rightarrow L_\Gamma$ is bijective.*

For our toils, we can now show that L_Γ satisfies the following defining universal property:

Theorem 5.23. *Let L be a Lie algebra, and suppose that $X = \{x_1, \dots, x_r\}$ is a subset of L that satisfies*

$$[x_i, x_j] = 0 \text{ when } [v_i, v_j] = 1.$$

Then there is a unique algebra homomorphism $\psi_X : L_\Gamma \rightarrow L$ such that

$$\psi_X(\mathbf{v}_i) = x_i \text{ for } 1 \leq i \leq r.$$

Proof. As L_Γ is generated by V , if such a map exists then it is unique. Let $\psi_X = \phi_X \ell^{-1}$. As ψ_X is a \mathbb{Z} -module morphism, we only need to check the bracket operation on the basis $\ell(LE(M))$ of L_Γ . Let $a, b \in LE(M)$ and without loss of generality suppose that $a < b$. By Proposition 5.21 there exists $I \subset LE(M)$ and a set of integers $\{\alpha_c\}_{c \in I}$ such that

$$\begin{aligned} [\ell(a), \ell(b)] &= \sum_{c \in I} \alpha_c \ell(c) \\ \text{and } [\phi_X(a), \phi_X(b)] &= \sum_{c \in I} \alpha_c \phi_X(c). \end{aligned}$$

Therefore

$$\begin{aligned} \psi_X([\ell(a), \ell(b)]) &= \psi_X\left(\sum_{c \in I} \alpha_c \ell(c)\right) \\ &= \sum_{c \in I} \alpha_c \psi_X \ell(c) \\ &= \sum_{c \in I} \alpha_c \phi_X(c) \\ &= [\phi_X(a), \phi_X(b)] \\ &= [\psi_X(\ell(a)), \psi_X(\ell(b))]. \end{aligned} \quad \square$$

6. THE ISOMORPHISM BETWEEN L_Γ AND THE LCS ALGEBRA OF A_Γ

The algebra L_Γ inherits a grading from $\mathcal{L}(U)$ by letting $L_{\Gamma,i} = L_\Gamma \cap \mathcal{L}(U)_i$. We note that

$$L_{\Gamma,i} = \langle \ell(a) : a \in LE(M), |a| = i \rangle.$$

Previously we defined \mathcal{C} and \mathcal{D} to be the linear filtrations of A_Γ given by the lower central series, and the central series $\{D_i\}$ given in section 4 respectively.

Lemma 6.1. *Let $X = \{v_i\gamma_1(A_\Gamma) : 1 \leq i \leq r\} \subset L_{\mathcal{C}}$. The algebra homomorphism $\psi_X : L_\Gamma \rightarrow L_{\mathcal{C}}$ given by Theorem 5.23 respects the gradings of L_Γ and $L_{\mathcal{C}}$.*

Proof. We show that $\psi_X(L_{\Gamma,k}) \subset L_{\mathcal{C},k}$ by induction on k . As $\psi_X(\mathbf{v}_i) = v_i\gamma_1(A_\Gamma)$, and $L_{\Gamma,1}$ is spanned by $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, the case $k = 1$ holds. For the inductive step, pick $a \in LE(M)$ such that $|a| = k$. Let $S(a) = (b, c)$ be the standard decomposition of a , with $|b| = i$, $|c| = j$, and $i + j = k$. Then by induction $\psi_X(\ell(b)) \in L_{\mathcal{C},i}$ and $\psi_X(c) \in L_{\mathcal{C},j}$, hence

$$\psi_X(\ell(a)) = [\psi_X(\ell(b)), \psi_X(\ell(c))] \in L_{\mathcal{C},i+j} = L_{\mathcal{C},k}.$$

As such elements span $L_{\mathcal{C},k}$, we have the required result. \square

By Proposition 4.11 we know that $\gamma_k(A_\Gamma) \subset D_k$ for all k . Hence by Proposition 2.6 the identity map $A_\Gamma \rightarrow A_\Gamma$ induces a graded algebra homomorphism $\alpha : L_{\mathcal{C}} \rightarrow L_{\mathcal{D}}$.

Lemma 6.2. *The mapping $gD_{k+1} \mapsto \mu(g)_k$ induces a graded algebra homomorphism $\beta : L_{\mathcal{D}} \rightarrow \mathcal{L}(U)$.*

Proof. D_{k+1} is the kernel of the homomorphism $D_k \rightarrow U_k$ given by $g \mapsto \mu(g)_k$. Therefore the induced map $\beta : L_{\mathcal{D}} \rightarrow \mathcal{L}(U)$ is well-defined. As $\mu(g)_k \in \mathcal{L}(U)_k$, this map also respects gradings. The fact that β is a homomorphism is implied by parts (1), (5) and (6) of Lemma 4.10. \square

We now have a chain of graded algebra homomorphisms

$$L_\Gamma \xrightarrow{\psi_X} L_{\mathcal{C}} \xrightarrow{\alpha} L_{\mathcal{D}} \xrightarrow{\beta} \mathcal{L}(U)$$

which allows us to prove the main theorem of this paper.

Theorem 6.3. *L_Γ , $L_{\mathcal{C}}$, and $L_{\mathcal{D}}$ are isomorphic as graded Lie algebras. Furthermore, the central filtrations \mathcal{C} and \mathcal{D} are equal, so that $\gamma_k(A_\Gamma) = D_k$ for all $k \geq 1$.*

Proof. We start by calculating the image of $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ under $\beta\alpha\psi_X$. We have

$$\begin{aligned} \beta\alpha\psi_X(\mathbf{v}_i) &= \beta\alpha(v_i\gamma_1(A_\Gamma)) \\ &= \beta(v_iD_1) \\ &= \mu(v_i)_1 \\ &= \mathbf{v}_i \end{aligned}$$

Therefore the image of $\beta\alpha\psi_X$ is L_Γ , and as $\beta\alpha\psi_X$ takes the generators to themselves, is the identity map on L_Γ . In particular, ψ_X must be injective. By Proposition 2.4, the algebra $L_{\mathcal{C}}$ is generated by the set X , hence ψ_X is also surjective, and is an isomorphism. We now know that $L_{\mathcal{C}}$ and L_Γ are isomorphic as graded Lie algebras. Then $\beta\alpha$ maps $L_{\mathcal{C}}$ isomorphically onto L_Γ , so the map α is also injective. Looking at each graded piece, each homomorphism

$$\gamma_k(A_\Gamma)/\gamma_{k+1}(A_\Gamma) \xrightarrow{\alpha_k} D_k/D_{k+1}$$

is injective. We shall use this to show that $\gamma_k(A_\Gamma) = D_k$ by induction on k , and this will complete the proof of the main theorem. Note that $D_1 = \gamma_1(A_\Gamma)$ by definition.

Suppose that $\gamma_k(A_\Gamma) = D_k$. Then α_k is also surjective, so is an isomorphism. If $g \in D_k = \gamma_k(A_\Gamma)$, then

$$\begin{aligned} g \in D_{k+1} &\iff gD_{k+1} = 1 && \text{in } D_k/D_{k+1} \\ &\iff \alpha_k^{-1}(gD_{k+1}) = 1 && \text{in } \gamma_k(A_\Gamma)/\gamma_{k+1}(A_\Gamma) \\ &\iff g\gamma_{k+1}(A_\Gamma) = 1 && \text{in } \gamma_k(A_\Gamma)/\gamma_{k+1}(A_\Gamma) \\ &\iff g \in \gamma_{k+1}(A_\Gamma). \end{aligned}$$

Hence $\gamma_{k+1}(A_\Gamma) = D_{k+1}$. \square

We conclude with an important consequence of Theorem 6.3 and Proposition 4.11:

Theorem 6.4. *If $k \in \mathbb{N}$, then $\gamma_k(A_\Gamma)/\gamma_{k+1}(A_\Gamma)$ is free-abelian, and $A_\Gamma/\gamma_k(A_\Gamma)$ is torsion-free nilpotent.*

Example 6.5. Let Γ be the small example graph given in Figure 1. We have already worked out the bracketing of Lyndon elements of length 3 in example 5.15. The isomorphism given in Theorem 6.3 tells us that $\gamma_3(A_\Gamma)/\gamma_4(A_\Gamma)$ is freely generated by $[v_1, [v_1, v_2]]\gamma_4(A_\Gamma)$, $[v_1, [v_1, v_3]]\gamma_4(A_\Gamma)$, $[[v_1, v_2], v_2]\gamma_4(A_\Gamma)$, $[[v_1, v_3], v_2]\gamma_4(A_\Gamma)$, and $[[v_1, v_3], v_3]\gamma_4(A_\Gamma)$.

REFERENCES

- [1] K.-T. Chen, R. H. Fox, and R. C. Lyndon. Free differential calculus. IV. The quotient groups of the lower central series. *Ann. of Math. (2)*, 68:81–95, 1958.
- [2] G. Duchamp and D. Krob. The lower central series of the free partially commutative group. *Semigroup Forum*, 45(3):385–394, 1992.
- [3] G. Duchamp and D. Krob. Free partially commutative structures. *J. Algebra*, 156(2):318–361, 1993.
- [4] Elisabeth R. Green. *Graph products of groups*. PhD thesis, The University of Leeds, 1990.
- [5] D. Krob and P. Lalonde. Partially commutative Lyndon words. In *STACS 93 (Würzburg, 1993)*, volume 665 of *Lecture Notes in Comput. Sci.*, pages 237–246. Springer, Berlin, 1993.
- [6] Pierre Lalonde. Bases de Lyndon des algèbres de Lie libres partiellement commutatives. *Theoret. Comput. Sci.*, 117(1-2):217–226, 1993. Conference on Formal Power Series and Algebraic Combinatorics (Bordeaux, 1991).
- [7] Pierre Lalonde. Lyndon heaps: an analogue of Lyndon words in free partially commutative monoids. *Discrete Math.*, 145(1-3):171–189, 1995.
- [8] Peter Linnell, Boris Okun, and Thomas Schick. The strong Atiyah conjecture for right-angled Artin and Coxeter groups. *arXiv:1010.0606*, 2010.
- [9] M. Lothaire. *Combinatorics on words*. Cambridge Univ Pr, 1997.
- [10] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory*. Dover Publications Inc., New York, revised edition, 1976. Presentations of groups in terms of generators and relations.
- [11] Richard D. Wade. Johnson homomorphisms and actions of higher-rank lattices on right-angled Artin groups. *arXiv:1101.2797*, 2011.

MATHEMATICAL INSTITUTE, 24-29 ST GILES', OXFORD OX1 3LB, UK
E-mail address: `wade@maths.ox.ac.uk`